Justifying Cursed Equilibria via Partial Awareness

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Abstract

We show that given any finite Bayesian game with a commonly-known common prior probability distribution, its set of cursed equilibria coincides a set of Bayesian Nash equilibria of an augmented game where players perceive other players types as if they are partially aware of others' original information structures. Consistent with the intuition that cursedness implies scarce computational resource, partial awareness is equivalent to a reduction of the complexity of players' strategic computation. This result also shows the potential of using unawareness to formulate imperfect strategic sophistication.

1 Introduction

In a game with incomplete information, if players have correct beliefs about other players' information and expect others rationally choose actions depending on their information, then the proper equilibrium concept is the standard Bayesian Nash equilibria. But when players are not sophisticated, one common mistake is to take other players' actions as unrelated with their private information. In reality, the way agents make decisions often lies between the two extremes. By a χ-weighted combination of the two situations, Eyster and Rabin (2005) introduce the concept of *cursed equilibria* to explain field or experimental data and do statistical inferences.

However, the notion of Bayesian Nash equilibrium itself is legitimate even when players have wrong (incompatible with the reality) or inconsistent (unable to be derived from a common prior) beliefs[1]. A large proportion of the economic literature assumes that prior beliefs must be common knowledge, so that there is no event that some players know could happen while some don't. However, sometimes we might need to relax this assumption to make the Bayesian Nash framework more flexible. In this paper, we want to show that by properly choosing more general beliefs, a cursed equilibrium can be justified as a Bayesian Nash equilibrium in a more general sense.

The equivalence is not obvious whenever the weight of cursedness χ is positive. In the proof of the existence of cursed equilibria, Eyster and Rabin (2005) let every player have two possible payoffs at every state. With probability $1 - \chi$, it depends on others' types, and with probability χ, it does not. In this virtual game, the Bayesian Nash equilibrium is a cursed equilibrium of the original game. Such formulation shows the existence of cursed equilibria but is not a very good justification within a Bayesian Nash framework, because it is difficult to argue that players' payoff functions are so versatile.

Using the idea of partial awareness, we want to justify cursed equilibrium within a Bayesian Nash framework . To see the intuition of awareness, suppose that a player's type is a state of mind determined by his information. In many realistic situations, the processing of the

[1]See Myerson (2004) for a discussion and Mertens and Zamir (1985) for a formal model.

information is not perfect. This information can be a multidimensional signal and the player, as a receiver of the signal, may lack the ability to either perceive, or measure, or understand the variations in certain dimensions. Li (2006) considers a model in which the player is "aware" of and actually uses only some dimensions of the signal. Taking a different analogy, Heifetz et al. (2006) say that all information is expressed in some language and to express complex information, a language must be rich in its expressive power. For example, a language with a restricted vocabulary in general has less expressive power. In this sense, they define a player's awareness by the expressive power of the language that he uses.

Heifetz et al. (2007) provide a general framework formulating unawareness in games with incomplete information. In this paper, we focus on a specific situation that players may be partially aware of others' types, in the sense that the perceived types can be represented by a partition and the original types can be represented by another however finer partition, on the same set of states.

It is not very surprising that partial awareness can explain some imperfect strategic behaviors. What we show is that under certain assumptions the equilibria derived from partially aware types are exactly equivalent to the cursed equilibria.

It takes two steps to show the result. First we modify every player's types. While keeping every type's belief about the exogenous parameter, we let this type perceive an augmented set of other players' types by adding a state where all other players do not have any information. By assuming that the type believes others to play an averaged strategy at this additional state, we can show that a Bayesian Nash equilibrium is exactly a cursed equilibrium. Second we show that the augmented set of types imply that the perceived information structure is always worse than the original information structure in the sense of Blackwell condition. Therefore using the result of Green and Stokey (1978), we can show that both the perceived information structure and the original information structure can be represented by two partitions of the same set of states with the same probability measure, and the partition representing the original information structure refines the other. This matches our definition of partial awareness.

A closely related work is Miettinen (2007). There he first defines the original set of states as a partition of an interval of measure one. Then he defines new partition of the interval. With the new partition, at every original state, he allows every player to be able to understand other players' type-dependent strategies with probability $1 - \chi$, and not able to do so with probability χ. Therefore at every original state, the expected payoff function is just like the one in the virtual game in Eyster and Rabin (2005). He uses this idea and the concept of analogy based expectation equilibrium to provide a learning foundation of cursed equilibrium.

Roughly speaking, the analogy based expectation equilibrium allows players to partition other players' types and assume the strategies to be based on the members of the partition instead of those types. Since partial awareness can be considered as one possible reason for players to partition in a certain way, two ideas are quite similar. However, since the new partition in Miettinen (2007) has more members than the number of original states, he concludes that when players are partially (but not fully) cursed, they use more complex strategic computations. But in our framework, the partition representing the perceived information structure is always coarser than the one representing the original information structure, hence the implication is the opposite.

The paper is organized as follows. We set up the framework and review Bayesian Nash Equilibrium and cursed equilibrium in Section 2. A justification with expanded type spaces is shown in Section 3. We show the main result on partial awareness in Section 4. During the process, an example about a lemon market is demonstrated. In Section 5, we discuss a lemon market model with partial awareness without referring to cursed equilibrium. Conclusions are in Section 6.

2 Bayesian Nash equilibria and cursed equilibria

The game is a finite static game with incomplete information denoted by $\left\langle \Theta, T_i; q; A_i; u_i \right\rangle_{i=1}^{N}$. The set of players is $\{1, 2, \ldots, N\}$. The exogenous parameter is $\theta \in \Theta$. The space of player types is $T \equiv \times_{i=1}^{N} T_i$. A common prior probability distribution q puts positive measure on every state in $\Theta \times T$. Type t_i's belief on the parameter and other players' types (θ, t_{-i}) is given by $q(\theta, t_{-i}|t_i)$.

An action of Player i is $a_i \in A_i$, and A_i is the action set. All players' action profile is a vector $a \in A \equiv \times_{i=1}^{N} A_i$. The action profile space A is assumed to be fixed for all states. Player i's payoff function is $u_i : A \times \Theta \to \mathbf{R}$. It is also assumed that this information is common knowledge.

A mixed strategy σ_i for Player i specifies a probability distribution over actions for each type, $\sigma_i : T_i \to \Delta(A_i)$. Let $\sigma_i(a_i|t_i)$ be the probability that type t_i plays action a_i. A strategy profile is $\sigma(t) \equiv \times_{i=1}^{N} \sigma_i(t_i) : T \to \Delta(A)$. Let A_{-i} be the set of action profiles for players other than i, σ_{-i} be the strategy profile of players other than i, and $\sigma_{-i}(a_{-i}|t_{-i})$ be the probability that types $t_{-i} \in T_{-i}$ plays actions a_{-i} under strategy $\sigma_{-i}(t_{-i})$.

We review the definitions of Bayesian Nash equilibrium (Harsanyi, 1967-1968) and cursed equilibrium.

Definition 1. *A strategy profile σ is a Bayesian Nash equilibrium if for each Player $i = 1, \ldots, N$, each type $t_i \in T_i$, and each a_i^* such that $\sigma_i(a_i^*|t_i) > 0$,*

$$a_i^* \in \arg\max_{a_i \in A_i} \sum_{\theta \in \Theta} \sum_{t_{-i} \in T_{-i}} q(\theta, t_{-i}|t_i) \times \sum_{a_{-i} \in A_{-i}} \sigma_{-i}(a_{-i}|t_{-i}) u_i(a_i, a_{-i}; \theta). \tag{1}$$

In a cursed equilibrium (Eyster and Rabin, 2005), given other players' strategy σ_{-i}, Player i mistakenly believes that with probability $\chi \in [0, 1]$ other players play mixed strategies regardless their types, and these strategies average their true strategies over their types, which is

$$\bar{\sigma}_{-i}(a_{-i}|t_i) \equiv \sum_{t_{-i} \in T_{-i}} \sum_{\theta \in \Theta} q(\theta, t_{-i}|t_i) \sigma_{-i}(a_{-i}|t_{-i}). \tag{2}$$

Definition 2. *A strategy profile σ is a χ-cursed equilibrium if for each Player i, each type $t_i \in T_i$, and each a_i^* such that $\sigma_i(a_i^*|\theta_i) > 0$,*

$$\begin{aligned} a_i^* \in \arg\max_{a_i \in A_i} \sum_{\theta \in \Theta} \sum_{t_{-i} \in T_{-i}} q(\theta, t_{-i}|t_i) \times \sum_{a_{-i} \in A_{-i}} & \left[\chi \bar{\sigma}_{-i}(a_{-i}|t_i) + (1-\chi) \sigma_{-i}(a_{-i}|t_{-i})\right] \\ & \times u_i(a_i, a_{-i}; \theta). \end{aligned} \tag{3}$$

When $\chi = 0$, χ-cursed equilibrium coincides with Bayesian Nash equilibrium. When $\chi = 1$, every player assumes that other players' strategy is completely unrelated with their types, namely players are *fully cursed*.

2.1 Lemon market: part 1

Consider the following example, taken from Eyster and Rabin (2005). There is a used car, a seller and a buyer. The exogenous parameter $v \in \{v_h, v_l\}$ determines the car's value. At state v_h, the value to the seller is $2,000$, the value to the buyer is $3,000$; at state v_l, the value to both is 0.

Ex ante, each state happens with probability $\frac{1}{2}$. Suppose the seller has a perfect signal $s \in \{g, b\}$ so that $Pr(v = v_h|s = g) = Pr(v = v_l|s = b) = 1$. The buyer has no information besides the prior probability distribution.

Then at a fixed price P, both sides are able to choose "deal" or "no deal". Trade happens only if both choose "deal".

Let $P = 1,000$. The seller sells only when $s = b$, and the buyer who knows this chooses "no deal". In the unique Bayesian Nash equilibrium, no trade happens.

In the cursed equilibrium, a χ-cursed buyer believes that with probability χ the seller sells with probability $\frac{1}{2}$ irrespective of the signal, the car's expected value is $3000[(1-\chi)0+\chi\frac{1}{2}] = 1500\chi$. Hence, a buyer cursed with $\chi > \frac{2}{3}$ will buy. Also, the seller's strategy, selling whenever $s = b$, after being averaged over his types, is consistent with the buyer's belief.

3 An intermediate alternative justification

We let players to have types different from T. There will be a type $y_i \in Y_i$ corresponding to every type t_i, so that y_i shares t_i's belief on the parameter, but he believes that there is a state with positive probability that none of other players have information.

To formalize, we need, for every Player i, one set of types: (Y_i, p_i) and $N-1$ sets of types: $(Y_j^i, p_j^i), \forall j \neq i, j \in \{1, \ldots, N\}$. We define the relation among the type sets by two bijective mappings: $f_i : Y_i \to T_i$ and $f_j^i : Y_j^i \backslash y_{jx}^i \to T_j$, where y_{jx}^i denotes a special type related to the cursedness. Here Y_i is the set of new types of Player i, with exactly the same number of elements of set T_i, or $|Y_i| = |T_i|$. Not knowing that the types of Player j are in Y_j, Player i believes that Y^i_j is the set of types of Player j, and $|Y^i_j| = |T_j| + 1$.

The following assumptions are made regarding Player i's prior belief p_i.

1) The marginal probability distributions of p_i about y_i is equal to that of q about $f_i(y_i)$.

$$\sum_{\theta\in\Theta}\sum_{y^i_{-i}\in Y^i_j} p_i(\theta, y_i, y^i_{-i}) = \sum_{\theta\in\Theta}\sum_{t_{-i}\in T_{-i}} q(\theta, f_i(y_i), t_{-i}); \tag{4}$$

2) The conditional probability distribution of p_i on (θ, y^i_{-i}) given y_i is

$$p_i(\theta, y^i_{-i}|y_i) = \begin{cases} (1-\chi)q(\theta, f^i_{-i}(y^i_{-i})|f_i(y_i)) & \text{if } y^i_{-i} \in \times_{j\neq i}(Y^i_j \backslash y^i_{jx}), \\ \chi \sum_{t_{-i}} q(\theta, t_{-i}|f_i(y_i)) & \text{if } y^i_{-i} = y^i_{-ix}, \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

Note that the second condition implies that Player i believes that either all other agents are cursed or none is cursed.

Player i also believes that j's belief p^i_j puts a positive measure on every state in $\Theta\times T_{-j}\times Y^i_j$: [2]

$$\sum_{\theta\in\Theta}\sum_{t_{-j}\in T_{-j}} p^i_j(\theta, t_{-j}, y^i_j) = \begin{cases} \chi & \text{if } y^i_j = y^i_{jx}, \\ (1-\chi)\sum_{\theta\in\Theta}\sum_{t_{-j}\in T_{-j}} q(\theta, t_{-j}, f^i_j(y^i_j)) & \text{if } y^i_j \neq y^i_{jx}. \end{cases} \tag{6}$$

And the conditional probability distribution of p_i on (θ, t_{-j}) given y^i_j is

$$p^i_j(\theta, t_{-j}|y^i_j) = \begin{cases} \sum_{t_j\in T_j} q(\theta, t_j, t_{-i}) & \text{if } y^i_j = y^i_{jx}, \\ q(\theta, t_{-j}|f^i_j(y^i_j)) & \text{if } y^i_j \in Y^i_j \backslash y^i_{jx}. \end{cases} \tag{7}$$

[2] Actually, this only matters in the next section, because to verify if a strategy profile is Bayesian Nash equilibria, only beliefs $\{p_i\}_{i=1}^N$ matters. We emphasize that with inconsistent beliefs it is generally not true that players could reason that others are rational. It is because players view others' information in a way that is inconsistent with what others believe.

For Player i, denote the strategy of other players by a function of the perceived types of others, namely $\sigma'_{-i} : Y^i_{-i} \to \Delta(A_{-i})$. We assume that when the state y^i_{-ix} happens, Player i believes that other players play averaged strategies.

Assumption 1. *For every Player i, every type y_i believes that given type profile y^i_{-ix}, all other players play the strategy*

$$\bar{\sigma}'_{-i}(a_{-i}|y_i) = \frac{1}{1 - Prob\{y^i_{-i} = y^i_{-ix}|y_i\}} \sum_{\theta\in\Theta} \sum_{y^i_{-i}\in Y^i_{-i}\setminus y^i_{-ix}} p_i(\theta, y^i_{-i}|y_i)\sigma'_{-i}(a_{-i}|y^i_{-i}).$$

Lemma 1. *With Assumption 1 hold, the augmented game's set of Bayesian Nash equilibria coincides the set of cursed equilibria of the original game.*

Proof. First by Equation 5, for every y_i,

$$Prob\{\tilde{y}^i_{-i} = y^i_{-ix}|\tilde{y}_i = y_i\} = \sum_{\theta\in\Theta} p_i(\theta, y^i_{-ix}|y_i) = \sum_{\theta\in\Theta} \chi \sum_{t_{-i}} q(\theta, t_{-i}|f_i(t_i)) = \chi.$$

Then by this and Equation 5, Assumption 1 implies that

$$\bar{\sigma}'_{-i}(a_{-i}|y_i) = \sum_{\theta\in\Theta} \sum_{y^i_{-i}\in Y^i_{-i}\setminus y^i_{jx}} q(\theta, f^i_j(y^i_{-i})|f_i(y_i))\sigma'_{-i}(a_{-i}|y^i_{-i}). \tag{8}$$

Second by Definition 1, in a Bayesian Nash equilibrium $\sigma'' : \times_{i=1}^N Y_i \to \Delta(A)$[3], for each Player $i = 1, \ldots, N$, each type $y_i \in Y_i$, and each a^*_i such that $\sigma''_i(a^*_i|y_i) > 0$,

$$a^*_i \in \arg\max_{a_i\in A_i} \sum_{\theta\in\Theta} p_i(\theta, y^i_{-ix}|y_i) \sum_{a_{-i}\in A_{-i}} \sigma'_{-i}(a_{-i}|y^i_{-ix})u_i(a;\theta) + \sum_{\theta\in\Theta} \sum_{y^i_{-i}\in Y^i_{-i}\setminus y^i_{-ix}} p_i(\theta, y^i_{-i}|y_i) \times \sum_{a_{-i}\in A_{-i}} \sigma'_{-i}(a_{-i}|y^i_{-i})u_i(a;\theta).$$

[3]Note that player j's true strategy $\sigma''_j(\cdot|y_j)$ is equivalent to $\sigma'_j(\cdot|y^i_j)$ when $f_j(y_j) = f^i_j(y^i_j)$.

The two components in the objective function are, first by Assumption 1 and Equation 5,

$$\sum_{\theta\in\Theta} p_i(\theta, y^i_{-ix}|y_i) \sum_{a_{-i}\in A_{-i}} \sigma'_{-i}(a_{-i}|y^i_{-ix})u_i(a;\theta) = \\ \sum_{\theta\in\Theta}\sum_{t_{-i}\in T_{-i}} \chi q(\theta, t_{-i}|f^i(y_i)) \sum_{a_{-i}\in A_{-i}} \bar{\sigma}'_{-i}(a_{-i}|y_i)u_i(a;\theta),$$

and secondly

$$\sum_{\theta\in\Theta}\sum_{y^i_{-i}\in Y^i_{-i}\backslash y^i_{-ix}} p_i(\theta, y^i_{-i}|y_i) \times \sum_{a_{-i}\in A_{-i}} \sigma'_{-i}(a_{-i}|y^i_{-i})u_i(a;\theta) = \\ \sum_{\theta\in\Theta}\sum_{y^i_{-i}\in Y^i_{-i}\backslash y^i_{-ix}} (1-\chi)q(\theta, f^i_{-i}(y^i_{-i})|f_i(y_i)) \times \sum_{a_{-i}\in A_{-i}} \sigma'_{-i}(a_{-i}|y^i_{-i})u_i(a;\theta).$$

Hence the objective function is equivalent to

$$\sum_{\theta\in\Theta}\sum_{y^i_{-i}\in Y^i_{-i}\backslash y^i_{-ix}} q(\theta, f^i_{-i}(y^i_{-i})|f_i(y_i)) \times \sum_{a_{-i}\in A_{-i}} \big[\chi\bar{\sigma}'_{-i}(a_{-i}|y_i) + \\ (1-\chi)\sigma'_{-i}(a_{-i}|y^i_{-i})\big] \times u_i(a;\theta).$$

Compare it with the objective function in Definition 3, and in particular, compare Equation 8 and Equation 2. We see that the characterizations of cursed equilibrium and Bayesian Nash equilibrium are equivalent. The sets of two must be identical. **QED.**

3.1 Lemon market: part 2

Let the buyer believe that there is a new signal $s' \in \{x, g, b\}$ with prior probability $Pr(s' = x) = \chi, Pr(s' = g) = \frac{1}{2}(1-\chi), Pr(s' = b) = \frac{1}{2}(1-\chi)$. The buyer believes that when the signal is g or b, it again conveys perfect information. But when it is x, the seller knows no information. This is

$$Pr(v = v_h|s' = g) = Pr(v = v_l|s' = b) = 1,$$

$$Pr(v = v_h|s' = x) = Pr(v = v_l|s' = x) = \frac{1}{2}.$$

Thus the buyer believes that the seller's expected value is 3000 at $s' = g$, 1000 at $s' = x$, and 0 at $s' = b$. The buyer expects that if the seller sells, the signal $s' \in \{x, b\}$; if the seller sells with .5 probability at $s' = x$, then the car's expected value conditional on a deal is

$$3000 \frac{Pr(v = v_h, s' = x)0.5 + Pr(v = v_h, s' = b)}{0.5Pr(s' = x) + Pr(s' = b)} = \frac{3000\chi.5^2}{0.5(1-\chi) + .5\chi} = 1500\chi.$$

The buyer buys if $1500\chi > 1000$, or $\chi \geq \frac{2}{3}$, which is equivalent to the result in the first part of the example.

4 Partial awareness

The previous justification relies on adding artificial states. From the viewpoint of a modeler, there could be some cognitive reason for the deviation of the perception of players. We want to apply the idea of awareness on the relation between the perceived types and the original types. It requires taking the two sets of types as players' information structures and representing them by information partitions on the same set of states with the same prior belief. We show that the relation between the partitions matches the definition of partial awareness, therefore by Lemma 1 the main result follows.

We first explain what we mean by an information partition that represents a player's information structure. Generally speaking, consider a decision maker uncertain about the value of parameter $\phi \in \Phi$, where $\Phi = \{\phi_1, \ldots, \phi_K\}$ is a finite set. Let the prior probability distribution be $r = \{r_1, \ldots, r_K\}$.

An information structure has two alternative formalizations.

1. A set of types Y, and a prior probability distribution λ on $\Phi \times Y$. This is denoted by (Y, λ).

2. A set X, a probability distribution μ on $\Phi \times X$, and a partition P on X. This is denoted by (X, μ, P).

Definition 3. *We say that* (X, μ, P) *represents* (Y, λ) *if there is a mapping:*

$$\tau : P \rightarrow Y$$

such that

1) for each $y \in Y$*, each* $s \in \tau^{-1}(y)$*, and all* $\phi \in \Phi$*,*

$$\frac{\mu(\{\phi\} \times s)}{\mu(\Phi \times s)} = \lambda(\phi | y),$$

2) for each $y \in Y$*,*

$$\sum_{s \in \tau^{-1}(y)} \mu(\Phi \times s) = \lambda(\Phi, y).$$

Now we define partial awareness.

Definition 4. *Player* i *is partially aware of Player* j*'s original information structure if for all of his types, there is a set* X_j*, a probability measure* μ_j *on* $\Theta \times X_j$*, and two partitions of* X_j*:* P_j *and* P'_j*, such that*

1) The information structure (X_j, μ_j, P_j) *represents player* j*'s original information structure* (T_j, q)*;*

2) The information structure (X_j, μ_j, P'_j) *represents player* j*'s information structure that* i *perceives, or* (Y^i_j, p^i_j)*;*

3) P_j *is a refinement of* P'_j*.*

Before we show the main result, the following definition and theorem are useful. Let (Y, q) and (Y', q') be two information structures about the value of same $\phi \in \Phi \equiv \{\phi_1, \ldots, \phi_K\}$, with $Y = \{y_1, \ldots, y_L\}$ and $Y' = \{y'_1, \ldots, y'_H\}$. Denote the conditional probability distribution $q(y|\phi_k)$, $k \in \{1, \ldots, K\}$, by a row vector π_k. Denote the conditional probability distribution $q'(y'|\phi_k)$ by a row vector π'_k.

Definition 5. *Two information structures* (Y, q) *and* (Y', q') *satisfy Blackwell's condition if* *and only if there exists a Markov matrix B such that*

$$\Pi' = \Pi B, \tag{9}$$

where $\Pi = (\pi_k(y_l))$, $l \in \{1, \ldots, L\}$ *and* $\Pi' = (\pi'_k(y'_h))$, $h \in \{1, \ldots, H\}$.

Theorem 1. Green and Stokey (1978) *If two information structures* (Y, q) *and* (Y', q') *satisfy Blackwell's condition 9, then there exists* (X, μ, P, P') *such that*

1. (X, μ, P) *represents* (Y, q)*;*

2. (X, μ, P') *represents* (Y', q')*;*

3. P *refines* P'*.*

The proof of the theorem is mainly about how to construct the elements (X, μ, P, P'). We will show that in the following subsection. But first we can present the following result.

Proposition 1. *Given that Player j's original types are* (T_j, q) *and Player i believes that Player j's types are* (Y_j^i, p_j^i)*, Player i is partially aware of Player j's original information structure.*

Proof. 1. Conditional on every $\omega \in \Theta \times T_{-j}$, Player j's original type t_j's probability distribution is $q(t_j|\omega)$. Let the set T_j be indexed by $m \in \{1, \ldots, |T_j|\}$ and the set $\Theta \times T_{-j}$ be indexed by $n \in \{1, \ldots, |\Theta \times T_{-j}|\}$. We define a matrix Π_j such that an element $\pi_{nm} = q(t_{jm}|\omega_n)$.

2. Conditional on every $\omega \in \Theta \times T_{-j}$, by Conditions 6 and 7, the probability of type y_j^i given ω is

$$p_j^i(y_j^i|\omega) = \begin{cases} \chi & \text{if } y_j^i = y_{jx}^i, \\ (1-\chi)q(t_j|\omega) & \text{if } y_j^i \neq y_{jx}^i, \text{ and } f_j^i(y_j^i) = t_j. \end{cases} \tag{10}$$

Let the set Y_j^i be indexed by $m' \in \{1, \ldots, |Y_j^i|\}$. Define a matrix Π_j^i such that an element $\pi'_{nm'} = p_j^i(y_{jm'}^i|\omega_n)$. Properly arrange the indexes we will have that

$$\Pi_j^i = \begin{pmatrix} \chi I_{|\Theta \times T_{-j}| \times 1}, & (1-\chi)\Pi_j \end{pmatrix}.$$

Therefore there is a Markov matrix

$$B = \begin{pmatrix} \chi & 1-\chi & 0 & \dots & 0 \\ \chi & 0 & 1-\chi & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \chi & 0 & 0 & \dots & 1-\chi \end{pmatrix},$$

such that $\Pi_j^i = \Pi_j B$. Blackwell's condition 9 is satisfied. By Theorem 1, the result follows. **QED.**

Putting Lemma 1 and Proposition 1 together, we have the main result.

Theorem 2. *For any $\chi \in (0, 1]$, the set of cursed equilibria of a game coincides a set of Bayesian Nash equilibria of an augmented game where players are partially aware of other players' original information structures.*

Define the complexity of his strategic computation by the cardinality of the partitions of other players perceived by Player i. Because partition P_j refines P'_j, when Player i perceives P'_j rather than P_j, the complexity decreases.

Corollary 1. *The complexity of players' strategic computation decreases when the cursedness χ becomes positive.*

4.1 Constructing information partitions.

The construction uses the method of Green and Stokey (1978) shown in the proof of Theorem 1. For notational brevity, we omit the subscripts of (X_j, μ_j, P_j, P'_j). The set $X = T_j \times Y_j^i$. The partitions $P = \{(t_j, y_j^i) | t_j \in T_j, y_j^i \in Y_j^i\}$ and $P' = \{T_j \times \{y_j^i\} | y_j^i \in Y_j^i\}$. We see that P refines P'.

The probability measure

$$\mu\big(\{(\omega, x)\}\big) = \mu\big(\{(\omega_n, t_{jm}, y_{jm'}^i)\}\big) = q(\omega_n, t_{jm}) b_{mm'}.$$

where $b_{mm'}$ is the element of Markov matrix B.

Now we need to show that (X, μ, P) represents (T_j, q), i.e. they satisfy Definition 3. Take $\tau((t_j, y_j^i)) = t_j$, so that $\tau^{-1}(\bar{t}_j) = \{\{(\bar{t}_j, y_j'^i)\}|y_j'^i \in Y_j^i\}$. Then

$$\frac{\mu(\omega_n, t_{jm}, y^i_{jm'})}{\mu(\Theta \times T_{-j} \times (t_{jm}, y^i_{jm'}))} = \frac{q(\omega_n, t_{jm})b_{mm'}}{\sum_n q(\omega_n, t_{jm})b_{mm'}} = q(\omega_n | t_{jm}).$$

This verifies the first condition in the definition.

Because $\sum_{m'} b_{mm'} = 1$ for every m,

$$\begin{aligned} \sum_{w \in \tau^{-1}(t_{jm})} \mu(\Theta \times T_{-j} \times w) &= \sum_n \sum_{m'} \mu(\omega_n, t_{jm}, y^i_{jm'}) \\ &= \sum_n \sum_{m'} q(\omega_n, t_{jm}) b_{mm'} \\ &= q(\Theta \times T_{-j}, t_{jm}). \end{aligned}$$

The second condition in the definition is also verified.

To show that (X, μ, P') represents (Y_j^i, p_j^i), define $\tau((t_j, y_j^i)) = y_j^i$ so that $\tau^{-1}(\bar{y}_j^i) = T_j \times \{\bar{y}_j^i\}$, then follow the same logic.

4.2 Lemon market: part 3

The probability distribution specified in part 2 implies that for the original types of the seller we have

$$\Pi = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and $\Pi_{vs} = Pr(s|v)$, where $v \in \{v_h, v_l\}$ is the row index and $s \in \{g, b\}$ is the column index.

For the perceived types of the seller, we have

$$\Pi' = \begin{pmatrix} \chi & 1-\chi & 0 \\ \chi & 0 & 1-\chi \end{pmatrix},$$

and $\pi'_{vs'} = Pr(s'|v)$, where $v \in \{v_h, v_l\}$ is the row index and $s' \in \{x, g, b\}$ is the column index.

The Markov matrix is $B = \Pi'$.

We construct a set $X = \{g, b\}\times\{x, g, b\}$ and partitions P and P', where $P = \{\{gg\}, \{gx\}, \{gb\}, \{bg\}, \{bx\}, \{bb\}\}$, and $P' = \{\{gg, bg\}, \{gx, bx\}, \{gb, bb\}\}$. It is easy to see that P refines P'. The measure μ is given as

$$\mu(v, s, s') = Pr(v) \times \Pi_{vs} \times \Pi'_{ss'}.$$

$\mu(v, s, s')$	$v = v_h$	$v = v_l$
$s = g, s' = x$	$\frac{\chi}{2}$	0
$s = g, s' = g$	$\frac{1-\chi}{2}$	0
$s = g, s' = b$	0	0
$s = b, s' = x$	0	$\frac{\chi}{2}$
$s = b, s' = g$	0	0
$s = b, s' = b$	0	$\frac{1-\chi}{2}$

Table 1: The probability distribution μ.

With a mapping $\tau(\{ss'\}) = s$, $\forall s \in \{g, b\}$ and $s' \in \{x, g, b\}$, we can check that (X, μ, P) is a representation of $s \in \{g, b\}$. A similar check for (X, μ, P') uses a mapping $\tau'(\{g, b\}\times\{s'\}) = s'$, $\forall s' \in \{x, g, b\}$.

5 An alternative lemon market model with partial awareness

To emphasize the idea we present this example in a reverse order. We define set $X = \{v_h, v_l\}\times\{g, b\}\times\{g', b'\}$, partition $P = \{\{gg'\}, \{gb'\}, \{bg'\}, \{bb'\}\}$, the partition $P' = \{\{gg', bg'\}, \{gb', bb'\}\}$. Hence P refines P'. Given $q \in (\frac{1}{2}, 1]$, the distribution μ is as

$\mu(v, s, s')$	$v = v_h$	$v = v_l$
$s = g, s' = g'$	$\frac{q}{2}$	0
$s = g, s' = b'$	$\frac{1-q}{2}$	0
$s = b, s' = g'$	0	$\frac{1-q}{2}$
$s = b, s' = b'$	0	$\frac{q}{2}$

Table 2: The probability distribution μ.

Thus if the buyer believes that the seller's information is represented by partition P', he knows that the seller could essentially have two types $s' \in \{g', b'\}$. Define a mapping

$\tau' : P' \to \{g', b'\}$ so that $\tau'(\{gg', bg'\}) = g'$ and $\tau'(\{gb', bb'\}) = b'$. It implies that

$$Pr(s' = g') = \sum_{v \in \{v_h, v_l\}} \sum_{s \in \{g, b\}} \mu(v, s, s' = g') = \frac{1}{2};$$

$$Pr(s' = b') = \frac{1}{2};$$

$$Pr(v = v_h | s' = g') = \frac{\frac{q}{2}}{\frac{1}{2}} = q;$$

$$Pr(v = v_l | s' = b') = \frac{\frac{q}{2}}{\frac{1}{2}} = q;$$

Believing this, the buyer expects that the seller never sells at type $s' = g'$. But if $s' = b'$, he sells because $2000(1-q) \leq 1000$, the expected value of the car is $3000(1-q)$ to the buyer. So if $q < \frac{2}{3}$, he wants to buy in equilibrium. It is just like in a cursed equilibrium.

But the original types are given by partition P.

Define a mapping $\tau : P \to \{g, b\}$ so that $\tau(\{ss'\}) = s$. It implies that for every $s'' \in \tau^{-1}(s)$,

$$Pr(s'' = gg') = \sum_{v \in \{v_h, v_l\}} \mu(v, s = g, s' = g') = \frac{q}{2};$$

$$Pr(s'' = gb') = \sum_{v \in \{v_h, v_l\}} \mu(v, s = g, s' = b') = \frac{1-q}{2};$$

$$Pr(s'' = bg') = \sum_{v \in \{v_h, v_l\}} \mu(v, s = b, s' = g') = \frac{1-q}{2};$$

$$Pr(s'' = bb') = \sum_{v \in \{v_h, v_l\}} \mu(v, s = b, s' = b') = \frac{q}{2};$$

and

$$Pr(v = v_h | s'' = gg') = Pr(v = v_h | s'' = gb') = 1;$$

$$Pr(v = v_l | s'' = bg') = Pr(v = v_l | s'' = bb') = 1.$$

These imply that the seller has perfect information. And the buyer shall not trade in the equilibrium.

In addition, the true types induce

$$\Pi = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

where $\pi_{11} = Pr(s = g|v = v_h)$, $\pi_{12} = Pr(s = b|v = v_h)$, $\pi_{21} = Pr(s = g|v = v_l)$, $\pi_{22} = Pr(s = b|v = v_l)$.

And the perceived types induce

$$\Pi' = \begin{pmatrix} q & 1-q \\ 1-q & q \end{pmatrix},$$

where $\pi'_{11} = Pr(s' = g|v = v_h)$, $\pi'_{12} = Pr(s' = b|v = v_h)$, $\pi'_{21} = Pr(s' = g|v = v_l)$, $\pi'_{22} = Pr(s' = b|v = v_l)$.

A Markov matrix $B = \Pi'$ satisfies Blackwell's condition $\Pi' = \Pi B$. The probability measure μ can be derived from Π and Π'.

6 Conclusions

In this paper we show the innovative cursed equilibrium concept is a special case of Bayesian Nash equilibrium with partial awareness. It justifies the first concept and suggests that the second concept is more general and may have potential to explain imperfect strategic sophistication from the cognitive aspects. Further applications in auction and trading can be promising.

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